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NOTE

LIST COLOURING WHEN THE CHROMATIC NUMBER IS CLOSE TO THE ORDER OF THE GRAPH

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Ohba has conjectured [7] that if G has $2\chi(G)+1$ or fewer vertices then the list chromatic number and chromatic number of G are equal. In this short note we prove the weaker version of the conjecture obtained by replacing $2\chi(G)+1$ by $\frac{5}{3}\chi(G)-\frac{4}{3}$.

1. Introduction

An instance of List Colouring consists of a graph G and a list $L(v)$ of colours for each vertex v of G. We are asked to determine if there is an *acceptable* colouring of G, that is a colouring in which each vertex receives a colour from its list, and no edge has both its endpoints coloured with the same colour. The *list-chromatic number* of G, denoted $\chi^l(G)$ is the minimum integer k such that for every assignment of a list $L(v)$ of size at least k to every vertex v of G, there exist an acceptable colouring of G. The list-chromatic number was introduced by Vizing [8], and independently by Erdős et al. [3]. This parameter has received a considerable amount of attention in recent years (see, e.g., [5], [1]).

Clearly, by definition, $\chi^l(G) \geq \chi(G)$ because $\chi(G)=k$ precisely if an acceptable colouring exists when each $L(v)$ is $\{1,\dots,k\}$. However, the converse inequality is not true, e.g., $\chi^l(K_{3,3})=3$ as can be easily verified by

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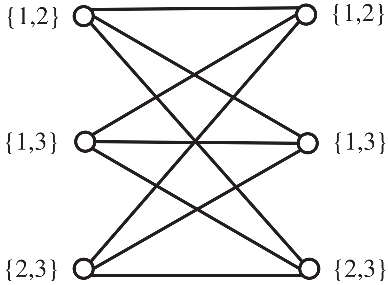

Figure 1. A bipartite graph with list chromatic number three

considering Figure 1. In fact, there are bipartite graphs with arbitrarily high chromatic number (indeed even for bipartite G, $\chi^l(G)$ is bounded from below by a function of the minimum degree which goes to infinity, see [1]). This shows that the gap between $\chi(G)$ and $\chi^l(G)$ can be arbitrarily large. Moreover it shows that $\chi^l(G)$ can not be bounded by any function of the chromatic number of G. This gives rise to the following intriguing question in the theory of Graph Colourings: Find conditions which guarantee the equality of chromatic and list-chromatic numbers.

There are many conjectures hypothesizing conditions on G which imply that $\chi(G)=\chi^l(G)$. Probably, the most famous of these is the List Colouring Conjecture (see [5]) which states that this is true if G is a line graph. One interesting example of a graph with $\chi=\chi^l$ was obtained in the original paper of Erdős et al. [3]. They proved that if G is complete k-partite graph with each part of size two then $\chi(G)=\chi^l(G)=k$. It took nearly twenty years until Ohba [7] noticed that this example is actually part of a much larger phenomenon. He conjectured (cf. [7]) that $\chi(G)=\chi^l(G)$ provided $|V(G)|\le 2\chi(G)+1$. In his paper Ohba proved this conjecture for graphs G with $|V(G)|\le\chi(G)+\sqrt{2\chi(G)}$. The conjecture was settled for some other special cases in [2]. Nevertheless, until the present there was no result which shows that $\chi(G)=\chi^l(G)$ for all graph with $|V(G)|\le\alpha\chi(G)$ with some α strictly bigger than 1. In this paper we obtain such a result. Our main theorem is as follows.

Theorem 1.1. $\chi^l(G)=\chi(G)$ *provided* $|V(G)|\le\frac{5}{3}\chi(G)-\frac{4}{3}$.

We prove this theorem via a probabilistic argument in the next section.

2. The proof of Theorem 1.1

In this section we prove our main result. First we need the following lemma which is of some independent interest.

Lemma 2.1. *For any integer t, if $\chi^l(G) > t$ then there exists a set of lists $L(v), v \in V(G)$ for which there is no acceptable colouring such that each list has at least t elements and the set $\mathcal{A} = \cup_{v \in V(G)} L(v)$ has size less than $|V(G)|$.*

Proof. Assume $\chi^l(G) > t$ and choose a set of lists $L(v), v \in V(G)$ for which there is no acceptable colouring, in which each list has size at least t and which minimize $|\mathcal{A}|$.

Now, if $|\mathcal{A}| < |V(G)|$ then we are done. So, we can assume the contrary. We consider the bipartite graph H with bipartition $(\mathcal{A}, V(G))$ and an edge between c and v precisely if $c \in L(v)$. Consider the smallest subset B of $\mathcal{A}$ which can not be covered by a matching in this graph. Clearly, B is non-empty and $|B| \leq |V| + 1$. Further, there is a matching of size $|B| - 1$ in H. If $|B| = |V| + 1$, this matching saturates V and points out an acceptable colouring for the List Colouring instance in which no colour is used more than once. Since, there is no such acceptable colouring, no such matching exists and so $|B| \leq |V|$. Now, by the minimality of B there is a matching M in H of size $|B| - 1$ whose endpoints in $\mathcal{A}$ are in B. Further, classical results in matching theory (see e.g. Theorem 1.1.3 of [6]) tell us that if W is the set of endpoints of M in V then for $v \notin W$, we have $L(v) \cap B = \emptyset$.

Let x be any vertex in $V(G) - W$ and replace $L(v)$ by $L(x)$ for every vertex $v \in W$. This yields a new List Colouring Problem in which the total number of colours in all lists is smaller than $|\mathcal{A}|$ (since all the new list are disjoint from B). Therefore by the minimality of our original choice, there exist an acceptable colouring of G for this new Lists Colouring instance. In particular this implies that we can obtain an acceptable colouring of $V(G) - W$ for the original lists $L(v)$. Since no colour in B is used in this colouring, using the colouring of W pointed out by M yields an extension of this colouring to a colouring of G in which no colour of B appears more than once. This contradicts our assumption that there is no acceptable colouring for this instance and proves the lemma. ∎

Proof of Theorem 1.1. We assume that the theorem is not true and obtain a contradiction. Let G be a counterexample to the theorem with as few vertices as possible and subject to this with as many edges as possible. Note that this implies that G is a complete $\chi(G)$-partite graph as adding an edge between vertices in different colour classes in an optimal colouring of G yields a new counterexample to the theorem. In the rest of the proof we refer to a colour class as a part to avoid confusion with the colours used in our acceptable colouring of G.

We claim that none of the parts in the partition of G have size two. To see this, assume the contrary and let $U=\{x,y\}$ be a part of size 2. Then the graph $G-U$ has chromatic number $\chi(G)-1$ and $|V(G)|-2$ vertices and therefore also satisfies

$$|V(G-U)| = |V(G)| - 2 \leq \left(\frac{5}{3}\chi(G) - \frac{4}{3}\right) - 2 = \frac{5}{3}\Big(\chi(G) - 1\Big) - \frac{5}{3} < \frac{5}{3}\chi(G-U) - \frac{4}{3}\,.$$

Hence by the minimality of G we obtain that $\chi^l(G–U)=\chi(G–U)=\chi(G)–1$. Now, by Lemma 2.1, there is an instance of List Colouring on G for which no acceptable colouring exists, in which each list has length at least $\chi(G) > |V(G)|/2$ and such that $|\mathcal{A}| < |V(G)|$. This implies that the set $L(x)\cap L(y)$ is non-empty. Let c be a colour in $L(x)\cap L(y)$. Since $\chi^l(G-U)=\chi(G)-1$, we know there is an acceptable colouring of $G-U$ from the lists $L(v)-c$. Colouring both x and y with c we obtain an extension of this colouring to an acceptable colouring of G from the original lists, a contradiction. So, there is indeed no part of size 2 in the partition.

Now, we let $k=\lfloor\frac{\chi(G)}{3}\rfloor$ and $r=\chi(G)–3k$. Thus, $r\in\{0,1,2\}$, each $L(v)$ has at least $3k+r$ elements, and $|V(G)|\leq\frac{5}{3}\chi(G)-\frac{4}{3}=5k+\frac{5}{3}r-\frac{4}{3}\leq 5k+r$. Denote by x the number of singleton parts in the partition of G. Since there are no parts of size 2, every non-singleton part has size at least 3. In addition the number of parts in this partition is $3k+r$. Therefore we obtain the following inequality $x+3(3k+r-x)\leq 5k+r$, which implies that $x\geq 2k+r$.

Let W be the union of some set of r singleton partition classes and note that we can obtain an acceptable colouring of W with the given lists greedily. Let T be the set of r colours used to colour W in one such acceptable colouring. Then to finish the proof of the theorem it is enough to prove the existence of an acceptable colouring of the $3k$-partite graph $G-W$ (which has at most $5k$ vertices) from the set of lists $\{L(v)-T\}$ (each of size $\geq 3k$). Indeed, by colouring W from T, we would extend the colouring of $G-W$ to an acceptable colouring of G from our original set of lists. This would be a contradiction. So we need only prove the lemma below.

Lemma 2.2. *Let $k\geq 1$ be an integer and let G be a $3k$-partite graph with at most $5k$ vertices and at least $2k$ parts of size one. Then*

$$\chi^l(G) \leq 3k.$$

Proof. Denote by $s_i, 1\leq i\leq 2k$ the vertices of $2k$ parts of G which have size one and denote by $U_j, 1\leq j\leq k$ the vertex sets of the remaining k parts in the partition. Note that, by definition, each U_i is an independent set. Let $\{L(v) \mid v \in V(G)\}$ be the set of lists of colours of size exactly $3k$ (if the original list has more than $3k$ colours we truncate it) assigned to the vertices of G. Our goal is to show that there exist an acceptable colouring of G from these lists. By Lemma 2.1 we can assume that the total number of colours in all lists is less than $|V(G)| \leq 5k$. Therefore for every pair of vertices $u \neq v$ we have that $|L(u) \cap L(v)| = |L(u)| + |L(v)| - |L(u) \cup L(v)| \geq 3k + 3k - 5k = k$.

Construct k disjoint sets of colours $C_1, \ldots, C_k$ each of size 3, as follows. First, for every $1 \leq i \leq k$ pick a colour c_i from $L(s_{2i-1}) \cap L(s_{2i})$, which is distinct from all previously chosen colours. Since, as we have already mentioned, $|L(s_{2i-1}) \cap L(s_{2i})| \geq k$, we can do this greedily. Next, for every $1 \leq j \leq 2k$ pick a colour c'_j form $L(s_j)$, such that it is distinct from all previously chosen ones and also from all the colours $\{c_1, \ldots, c_k\}$. Note that the size of each list $L(v)$ is $3k$ and that we are picking only $3k$ colours. Therefore during this process we will never run out of colours. Finally, let $C_i = \{c_i, c'_{2i-1}, c'_{2i}\}$ and let $\mathcal{C} = \cup_i C_i$. Note that by our construction the sets $C_1, \ldots, C_k$ have the following property: For any pair of colours from C_i there is always an acceptable colouring of the vertices s_{2i-1} and s_{2i} which uses this pair.

We will colour G in two rounds. In the first round, for every $1 \leq i \leq k$ pick a colour $t_i \in C_i$ uniformly at random and also pick a random permutation $\sigma : [k] \rightarrow [k]$. We colour vertices s_{2i-1}, s_{2i} using the colours in $C_i - \{t_i\}$ and also colour by $t_{\sigma(i)}$ all vertices in U_i which contain this colour in their list. Denote by V' the set of uncoloured vertices at the end of round one. Remove all the colours in $\mathcal{C}$ from the lists of vertices in V' and denote by $L'(v)$ the new list of colours. Note that, by definition, $L'(v) = L(v) - \mathcal{C}$ for every $v \in V'$. It is easy to see that if $L(v) = \mathcal{C}$ then v will always be coloured in the first round, therefore all vertices in V' have non-empty lists. Also note that the main property of this colouring procedure is that for every fixed $1 \leq i \leq k$ and for every colour $c \in \mathcal{C}$, the probability that $t_{\sigma(i)} = c$ is exactly $(1/3)(1/k) = 1/3k = 1/|\mathcal{C}|$.

For every vertex $v \in \cup_i U_i$, let x_v be a random variable which equals $1/|L'(v)|$ if v is still uncoloured after round one and zero otherwise. It is easy to see that if vertex $v \in U_i$ has a list $L'(v)$ of size t, then $|L(v) \cap \mathcal{C}| = |\mathcal{C}| - t$ and therefore the probability that v remains uncoloured in the end of round one, i.e. $v \in V'$, is

$$Pr\left(t_{\sigma(i)} \notin L(v) \cap \mathcal{C}\right) = 1 - Pr\left(t_{\sigma(i)} \in L(v) \cap \mathcal{C}\right) = 1 - \frac{|L(v) \cap \mathcal{C}|}{|\mathcal{C}|}$$
$$= 1 - \frac{|\mathcal{C}| - t}{|\mathcal{C}|} = \frac{t}{|\mathcal{C}|}.$$

Hence we can bound the expected value of $\sum_{v\in\cup_i U_i} x_v$ as follows

$$E\left(\sum_{v\in\cup_i U_i} x_v\right) = \sum_{v\in\cup_i U_i} E[x_v] = \sum_{t\geq 1}\sum_{v,\,|L'(v)|=t}\frac{1}{t}Pr\left(v\in V'\right)$$
$$=\sum_{t\geq 1}\sum_{v,\,|L'(v)|=t}\frac{1}{t}\,\frac{t}{|\mathcal{C}|} = \sum_{v\in\cup_i U_i}\frac{1}{|\mathcal{C}|} = \frac{|\cup_i U_i|}{|\mathcal{C}|} \leq \frac{5k-2k}{3k} = 1.$$

Therefore there exists a particular choice of colours t_i and of permutation σ that satisfies $\sum_v x_v \leq 1$. Fix such a t_i and σ together with the partial colouring produced in the first round. Then we claim that we can colour the vertices in V' greedily and thereby obtain an acceptable colouring of G. Indeed, put the vertices in V' in the increasing order of the sizes of their lists $L'(v)$. Assign colours to the vertices one by one in this order. We always colour a vertex by a colour in its list which is different from the colours used on the previous vertices. This process constructs an acceptable colouring and colours all the vertices of V'. If not, there exists a vertex $v\in V'$ with a list of size t_0 such that all the colours on $L'(v)$ were used to colour previous vertices. This implies that in our greedy colouring process there were at least t_0 vertices before v. By definition, all these vertices also have lists of size at most t_0. Altogether there are at least t_0+1 vertices in V' with the lists of size at most t_0. This contradicts our assumption that $\sum_v x_v \leq 1$, since

$$\sum_{v\in\cup_i U_i} x_v = \sum_{t\geq 1}\sum_{v\in V',\,|L'(v)|=t}\frac{1}{t} \geq \frac{t_0+1}{t_0} > 1.$$

This completes the proof of the lemma and the proof of Theorem 1.1. ∎ ∎

Remarks.

- It is worth mentioning that one can actually efficiently find the acceptable colouring guaranteed to exist by Lemma 2.2 using the method of conditional expectations due to Erdős and Selfridge ([4]). We omit the details.
- After this paper was written we noticed that our arguments together with some additional ideas can be also used to prove that Ohba's conjecture is asymptotically correct. Since the new proof is fairly involved and does not imply Theorem 1.1 it will appear in a separate paper.

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